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STATISTICS AND PROBABILITY
www.stt.msu.edu/~lepage
click on STT200_Sp09

Lecture Outline, 3-27-09 and 3-30-09. See pp. 436-444.

RANDOM VARIABLE Chapter 16

boats sold	probability
2	0.2
3	0.2
4	0.3
5	0.1
6	0.1
7	0.05
8	0.05
total	1

P(fewer than 3.7) = .4

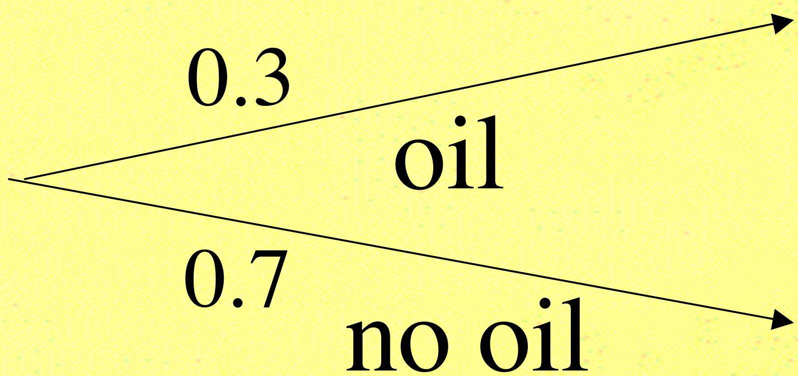
P(4 to 7) = .55

OIL DRILLING EXAMPLE

P(oil) = 0.3

Cost to drill 130
Reward for oil 400

net return
"just drill"



-130 + 400 = 270
drill oil

drill no oil
-130 + 000 = -130

A random variable is just a **numerical function** over the outcomes of a probability experiment.

EXPECTATION

Definition of E X

E X = sum of value times probability x p(x).

Key properties

E(a X + b) = a E(X) + b

E(X + Y) = E(X) + E(Y) (<u>always</u>, if such exist)

a. E(sum of 13 dice) = 13 E(one die) = 13(3.5).

b. E(0.82 Ford US + Ford Germany - 20M)

= 0.82 E(Ford US) + E(Ford Germany) - 20M

regardless of any possible dependence.

total of 2 dice

(3-15) of text

total of 2 dice	probability	product
2	1/36	2/36
3	2/36	6/36
4	3/36	12/36
5	4/36	20/36
6	5/36	30/36
7	6/36	42/36
8	5/36	40/36
9	4/36	36/36
10	3/36	30/36
11	2/36	22/36
12	1/36	12/36
sum	1	252/36 = 7

E (total) is just twice the 3.5 avg for one die

E(total) ← 7

(3-17 of text)

boats/month	probability	product
2	0.2	0.4
3	0.2	0.6
4	0.3	1.2
5	0.1	0.5
6	0.1	0.6
7	0.05	0.35
8	0.05	0.4
total	1	4.05

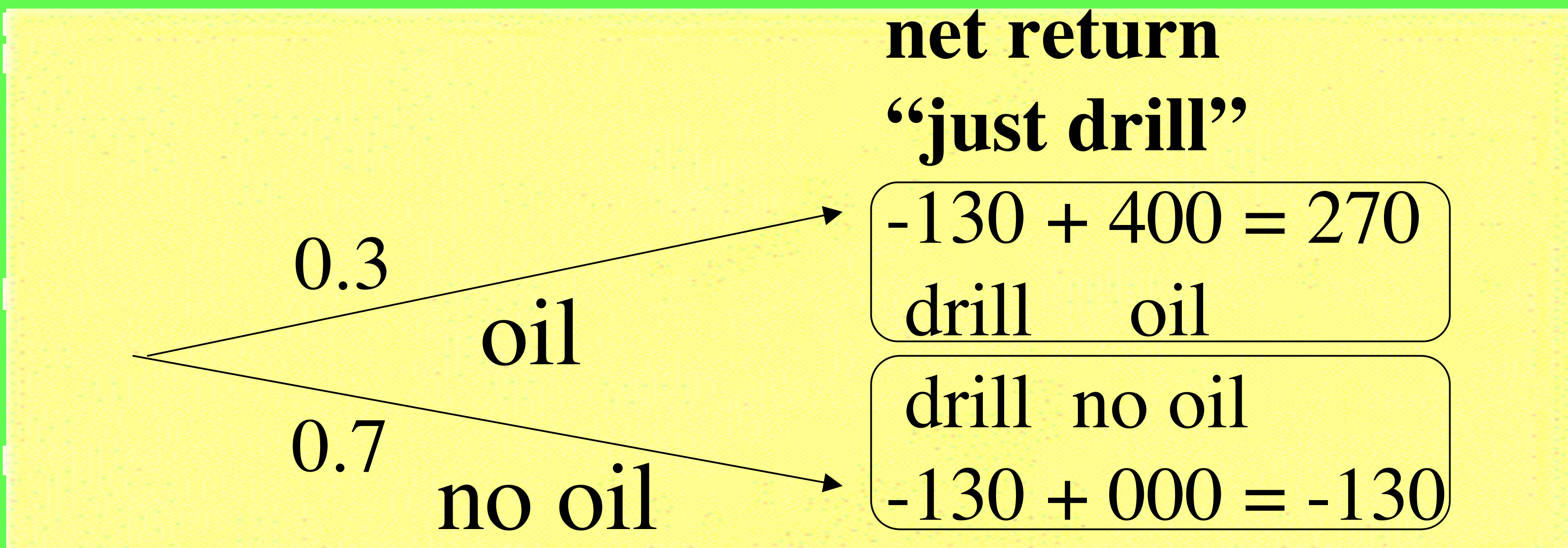
we avg 4.05 boats per month

E(number of boats this month)

OIL DRILLING EXAMPLE

P(oil) = 0.3

Cost to drill 130
Reward for oil 400



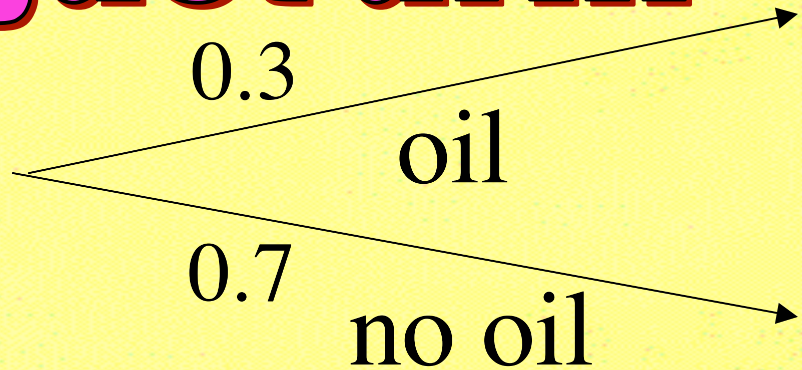
A random variable is just a **numerical function** over the outcomes of a probability experiment.

EXPECTATION IN THE OIL EXAMPLE

Expected return from policy "**just drill**" is the probability weighted average (NET) return

E(NET) = (0.3) (270) + (0.7) (-130) = 81 - 91 = -10.

just drill



net return from policy"just drill."

-130 + 400 = 270

drill oil

drill no-oil

-130 + 0 = -130

E(X) = -10

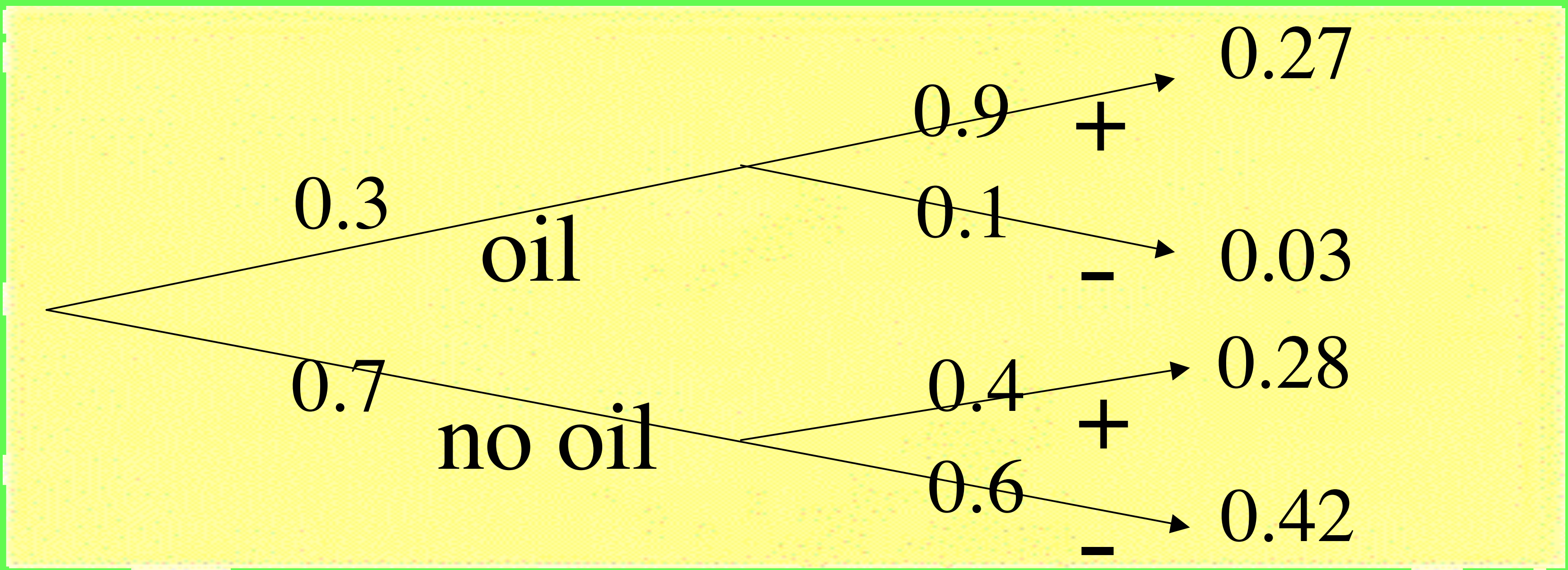
OIL EXAMPLE WITH A "TEST FOR OIL"

"costs"
TEST 20
DRILL 130
OIL 400

A test costing 20 is available.
This test has:

$P(\text{test} + \mid \text{oil}) = 0.9$

$P(\text{test} + \mid \text{no-oil}) = 0.4.$



Is it worth 20 to test first?

EXPECTED RETURN IF WE "TEST FIRST"

	net return	prob	prod
oil+	= -20 -130 + 400 = 250	0.27	67.5
oil-	= -20 - 0 + 0 = - 20	.03	- 0.6
no oil+	= -20 -130 + 0 = -150	.28	- 42.0
no oil-	= -20 - 0 + 0 = - 20	.42	- 8.4
	total	1.00	**16.5**

drill only if the test is +

E(NET) = .27 (250) - .03 (20) - .28 (150) - .42 (20)

= 16.5 (for the "test first" policy).

This average return is much preferred over the E(NET) = -10 of the "just drill" policy.

Variance and s.d. of boats/month

(3-17) of text

x	$p(x)$	$x\,p(x)$	$x^2\,p(x)$	$(x-4.05)^2\,p(x)$
2	0.2	0.4	0.8	0.8405
3	0.2	0.6	1.8	0.2205
4	0.3	1.2	4.8	0.0005
5	0.1	0.5	2.5	0.09025
6	0.1	0.6	3.6	0.38025
7	0.05	0.35	2.45	0.435125
8	0.05	0.4	3.2	0.780125
total	1.00	4.05	19.15	2.7475
quantity		$E\,X$	$E\,X^2$	$E\,(X - E\,X)^2$
terminology		mean	mean of squares	variance = mean of sq dev

s.d. = root(2.7474) = root(19.15 - 4.05^2) = 1.6576

VARIANCE AND STANDARD DEVIATION

Var(X) $=^{def}$ E (X - E X)^2 $=^{comp}$ E (X^2) - (E X)^2
i.e. Var(X) is the expected square deviation of r.v. X from its own expectation.

Caution: The computing formula (right above), although perfectly accurate mathematically, is sensitive to rounding errors.

Key properties:

Var(a X + b) = a^2 Var(X) (b has no effect).

sd(a X + b) = |a| sd(X).

VAR(X + Y) = Var(X) + VAR(Y) if X ind of Y.

EXPECTATION AND INDEPENDENCE

Random variables X, Y are INDEPENDENT if

$p(x, y) = p(x)\,p(y)$ for all possible values x, y.

If random variables X, Y are INDEPENDENT

$E(X\,Y) = (E\,X)\,(E\,Y)$ echoing the above.

$Var(X + Y) = Var(X) + Var(Y)$.

PRICE RELATIVES

Venture one returns random variable X per $1 investment. This X is termed the "price relative." This random X may in turn be reinvested in venture two which returns random random variable Y per $1 investment. The return from $1 invested at the outset is the product random variable XY.

EXPECTED RETURN

If INDEPENDENT, E(X Y) = (E X) (E Y).

PARADOX OF GROWTH

EXAMPLE:

\$1 → X

x	p(x)	x p(x)
0.8	0.3	0.24
1.2	0.5	0.60
1.5	0.2	<u>0.30</u>

E(X) = 1.14

WE AVERAGE 14% PER PERIOD

BUT YOU WILL NOT EARN 14%. Simply put, the average is not a reliable guide to real returns in the case of exponential growth.

EXPECTATION governs SUMS but sums are in the exponent

EXAMPLE:

\$1 → X

x	p(x)	$\text{Log}_e[x]\, p(x)$
0.8	0.3	-0.029073
1.2	0.5	0.039591
1.5	0.2	0.035218

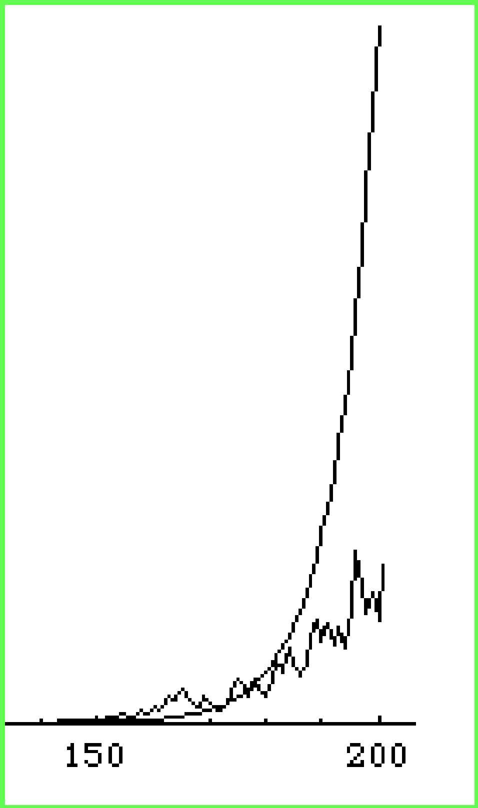
$E\ \text{Log}_e[X] = 0.105311$

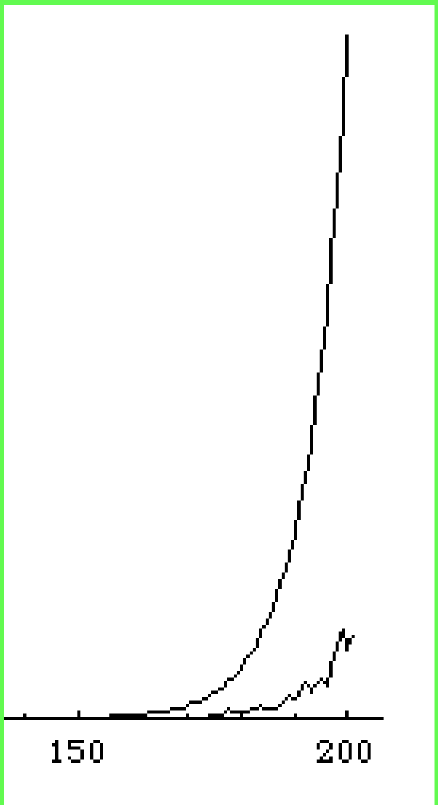
$$e^{0.105311..} = 1.11106..$$

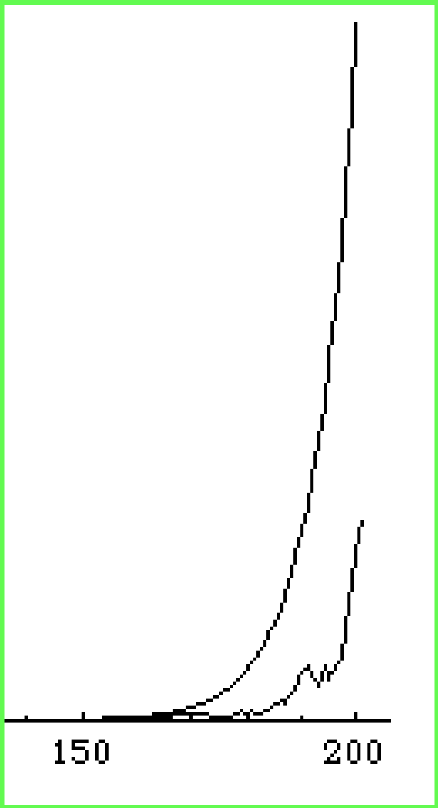
With INDEPENDENT plays your RANDOM return will compound at 11.1% not 14%.

(more about this later in the course)

COMPARING 1.14^n WITH THREE RANDOM EVOLUTIONS







you can see that 14% exceeds reality

Poisson Distribution Governing Counts of Rare Events

$p(x) = e^{-mean}\ mean^{x} / x!$

for x = 0, 1, 2, ..ad infinitum

Poisson
first best thing:

THE FIRST BEST THING ABOUT THE POISSON IS THAT THE MEAN ALONE TELLS US THE ENTIRE DISTRIBUTION!

note: Poisson sd = root(mean)

Poisson Cookies
400 raisins
144 COOKIES
mix well

E X = 400/144 ~ 2.78 **raisins per cookie**

sd = root(mean) = 1.67

(for Poisson)

Poisson Cookies

e.g. X = number of raisins in MY cookie. Batter has 400 raisins and makes 144 cookies.

$E X = 400/144 \sim 2.78$ **per cookie**

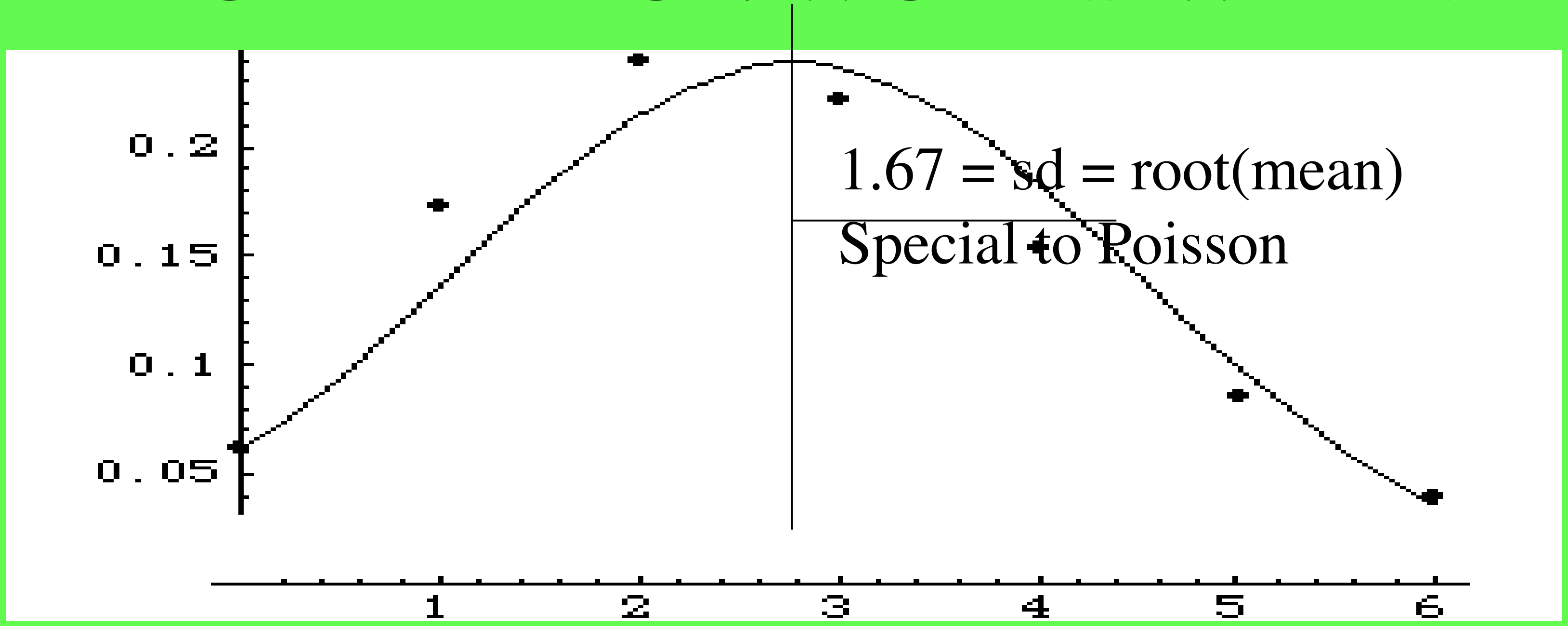
$p(x) = e^{-mean} \, mean^{x} / x!$

e.g. $p(2) = e^{-2.78} \, 2.78^{2} / 2! \sim 0.24$

(around 24% of cookies have 2 raisins)

Poisson second best thing

THE SECOND BEST THING ABOUT THE POISSON IS THAT FOR A MEAN AS SMALL AS 3 THE NORMAL APPROXIMATION WORKS WELL.



mean 2.78

Poisson at cards

e.g. X = number of times ace of spades turns up in 104 independent tries (i.e. from full deck)

X~ Poisson with mean 2

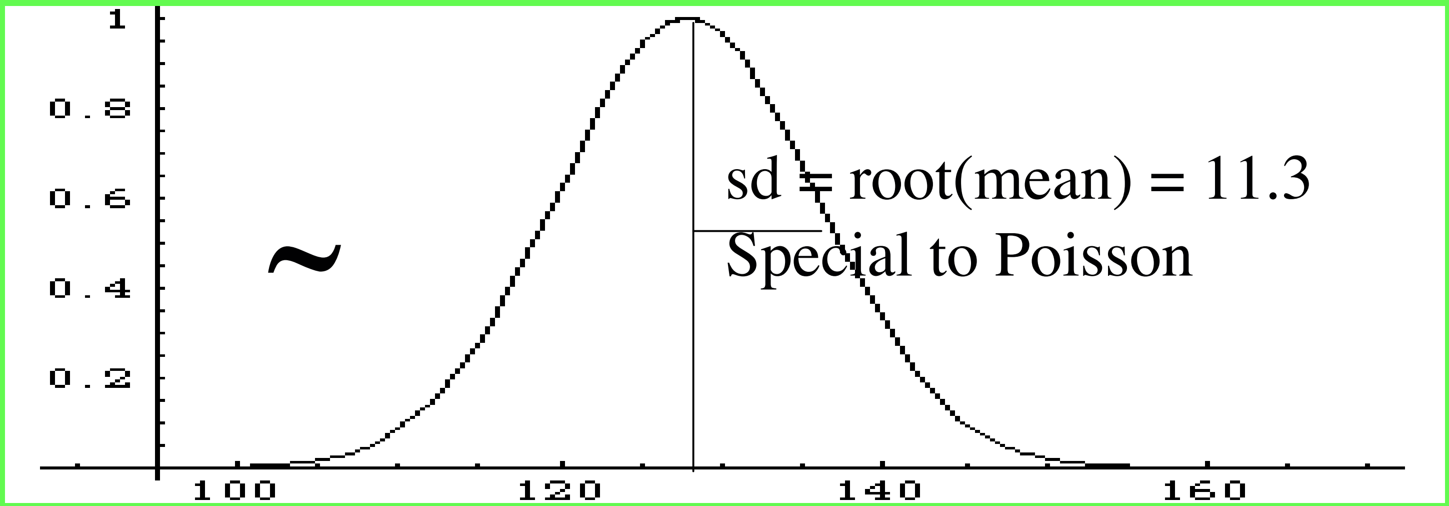
$p(x) = e^{-mean}\ mean^{x} / x!,\ x=0\ ..$

$p(3) = e^{-2}\ 2^{3} / 3! \sim 0.182205$

Poisson in Risk AVERAGE 127.8 ACCIDENTS PER MO.

E X = 127.8 accidents

If Poisson then sd = root(127.8) = 11.3049 and the approx dist is:



mean 127.8 accidents

Binomial

X = # of successes, n independent tries

e.g. X = number of times ace of spades turns up in 104 deals of 1 card from a shuffled full deck.

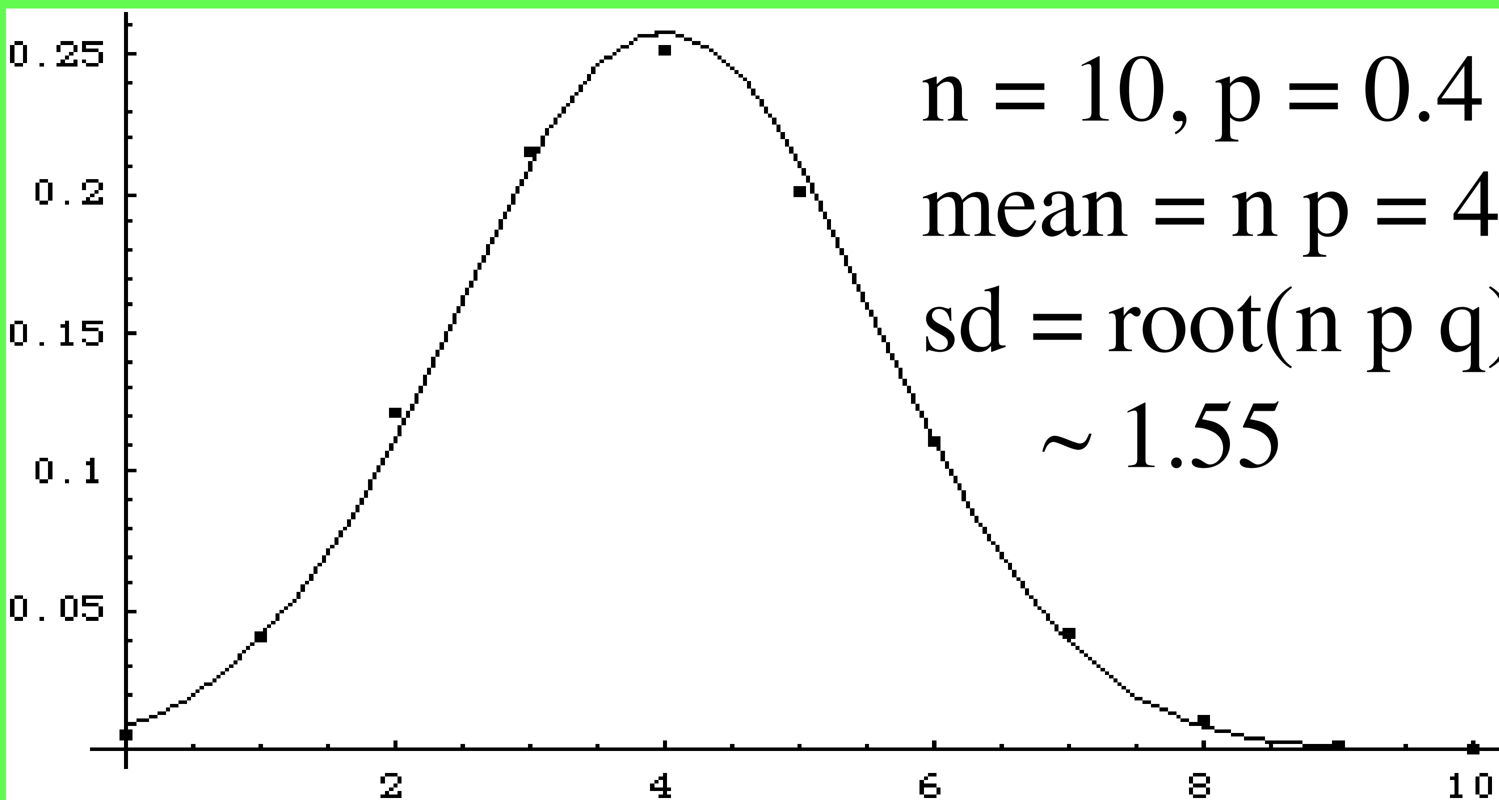
Binomial (n=104, p = 1/52)

$p(x) = nCx^{*}\ p^{x}q^{n-x}$, n = 0 to n.

$p(3) = ((104!)/(3!\ 101!))\ (1/52)^{3}(51/52)^{49} \sim 0.182205$

Agrees with Poisson approximation of binomial!

Normal Approx of Binomial



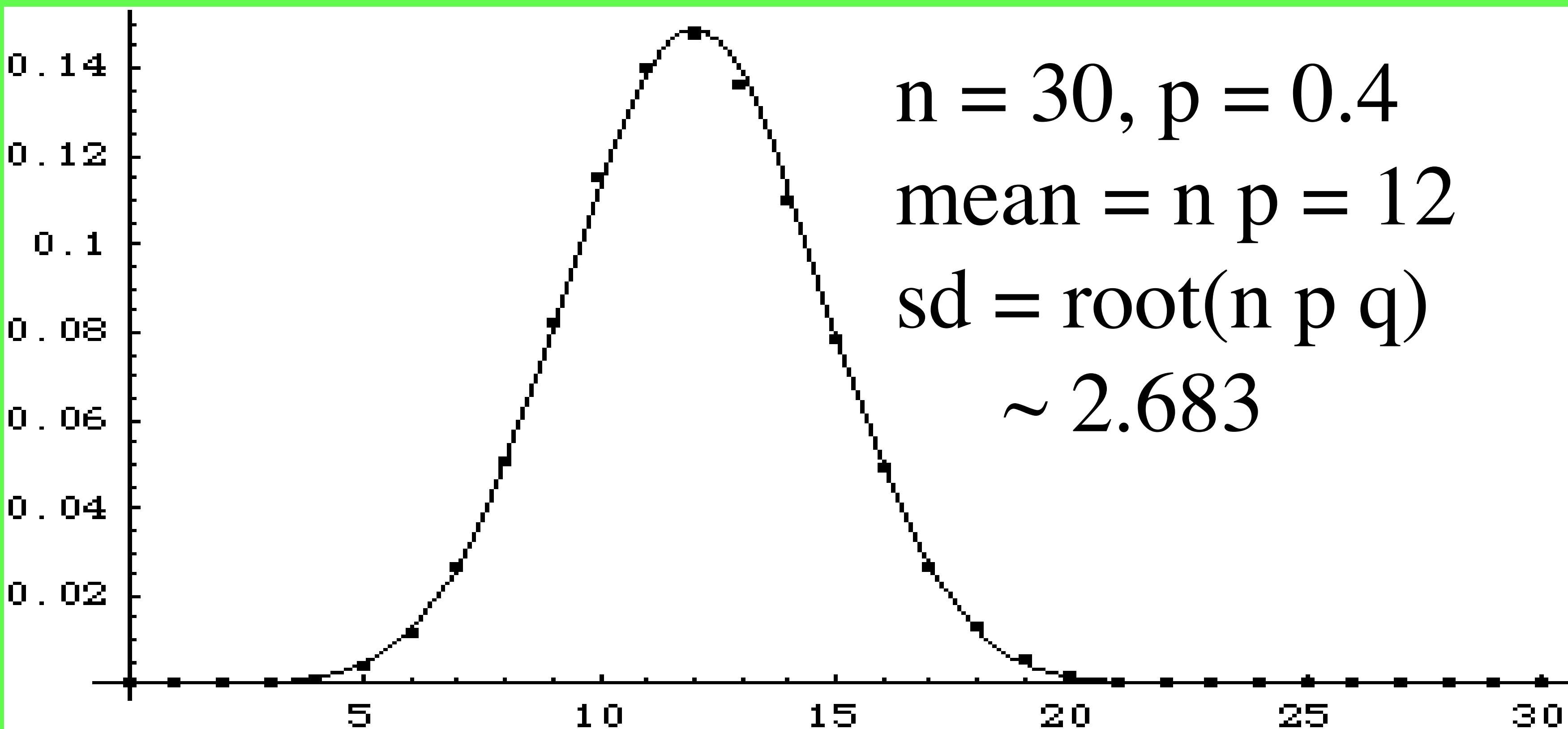
n = 10, p = 0.4

mean = n p = 4

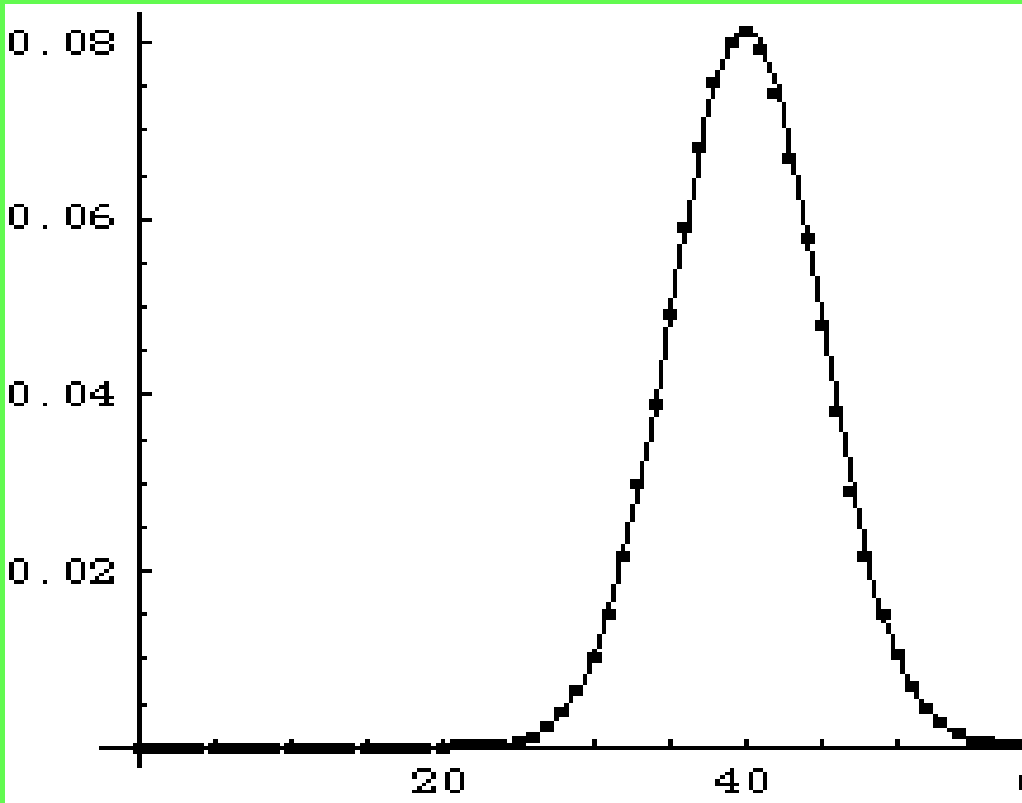
sd = root(n p q)

~ 1.55

Normal Approx of Binomial



Normal Approx of Binomial



n = 100, p = 0.4

mean = n p = 40

sd = root(n p q)

~ 4.89898